Contemporary Mathematics

A Counterexample to a conjecture of Bosio and Meersseman

David Allen and José La Luz

ABSTRACT. In [**BM**] the following is conjectured: If P is dual neighborly, then Z_P is diffeomorphic to the connected sum of products of spheres. In this paper a counterexample is provided.

Introduction

In [**A**] methods from unstable homotopy theory were used to extend homotopy computations for various spaces that are studied in Toric Topology. In that paper, one of the main theorems shows that through a range the higher homotopy groups of the Borel space are isomorphic to the higher homotopy groups of a certain wedge of spheres. The range restriction arises from the combinatorics of P and depends on the monomials that appear in the quotient in the face ring $\mathbb{Z}(P)$. The calculations in that paper provide enough homotopy theoretic information that allows for one to compare $\pi_*(Z_P)$ with $\pi_*(B_TP)$ via the fibration $Z_P \to B_TP \to BT^m$ to show that the conjecture can not hold. The computations are rather straightfoward, however, they illustrate a particularly useful application of 3.4 when it comes to answering or at least testing problems of this type. In Toric Topology it is well known by the work of Buchstaber, Panov and their collaborators that the higher homotopy groups of the moment angle complex Z_P are isomorphic to the higher homotopy groups of the complement of a complex coordinate subspace arrangement, sometimes denoted in the literature as $U(K)$. In [**BM**] the "link" X that appears in their conjecture has the same higher homotopy groups as the spaces that appear in Toric topology, in particular, the moment angle complex Z_P as well as the Borel space B_TP. This is pointed out by [**BM**] in the proof of 7.6 and 7.7.

The paper is set up as follows. In §1 the combinatorics needed in the sequel will be discussed. In §2 the main definitions from Toric Topology will be listed. In §3 theorems from homotopy theory will be listed and in §4 the main result will be proven.

2000 *Mathematics Subject Classification.* Primary:14M25; Secondary 57N65.

Key words and phrases. Toric Manifolds, Toric Topology, higher homotopy groups, homotopy type, moment angle complexes.

1. Combinatorics

All of the definitions in this section can be found in [**BP, B**]. For a more in-depth discussion of the material in this section the reader is urged to refer to one of these references.

Definition 1.1. The *affine hull* of the points $x_1, .., x_n$ where $x_i \in \mathbb{R}^n$ is the set $\{\sum_{i=1}^n a_i x_i | a_i \in \mathbb{R}, a_1 + \cdots + a_n = 1\}$.

A *convex polytope* is the convex hull of finitely many points in $\mathbb{R}^n$. We assume that all polytopes P contains 0 in its interior. $(\mathbb{R}^n)^*$ will denote the vector space dual to $\mathbb{R}^n$. There is an equivalent definition.

Definition 1.2. A convex polyhedron P is an intersection of finitely many half-spaces in $\mathbb{R}^n$.

$$P = \{\overrightarrow{x} \in \mathbb{R}^n | \langle \overrightarrow{l_i}, \overrightarrow{x} \rangle \geq -a_i, i = 1, .., m\}$$

The dimension of P is the dimension of its affine hull. A *supporting hyperplane* H of P is an affine hyperplane which intersects P such that P is contained in one the closed half-spaces determined by H. A *face* of P is $P \cap H$ where H is a supporting hyperplane. The *boundary* of P, denoted ∂P is the union of all the proper faces of P. The vertices of P are the 0-dimensional faces. The *facets* are the $(n-1)$-dimensional faces. That is, the co-dimension 1-faces. An n-dimensional convex polytope is *simple* if the number of facets meeting at each vertex is exactly n. For each co-dimension k face $F_k = \bigcap_{j=1}^k F_{\iota_j}$. P is said to be *q-neighborly* if $\bigcap_{j=1}^q F_{\iota_j} \neq \emptyset$ for any q facets. For example, the square is 1 neighborly. An n-dimensional convex polytope is called *simplicial* if there are at least n facets meeting at each vertex.

Definition 1.3. For any convex polytope $P \subset \mathbb{R}^n$ define its polar set $P^* \subset (\mathbb{R}^n)^*$ by

$$P^* = \{\mathbf{x}' \in (\mathbb{R}^n)^* | \ \langle \mathbf{x}', \mathbf{x} \rangle \geq -1 \ \forall \ \mathbf{x} \in P\}$$

P^* is a convex polytope since $0 \in P$. We recall some properties of cyclic polytopes [**BP, B**]. For $d \geq 2$ the moment curve M_d in $\mathbb{R}^d$ is the curve parametrized by $t \mapsto x(t) = (t, t^2, .., t^d), t \in \mathbb{R}^d$. A *cyclic polytope* of type $C(n, d)$ where $n \geq d+1$ and $d \geq 2$ is the convex hull of $\{x(t_1), .., x(t_n)\}$ where the t_i are distinct real numbers.

It is well known that a simplicial polytope is *k neighborly* if any k vertices span a face [**BP**]. We say a simplicial n polytope with an arbitrary number of vertices is *neighborly* if it is $[\frac{n}{2}]$ neighborly. If P is a neighborly polytope, then its dual P^* is said to be *dual neighborly.* In what follows we need to describe what sets of vertices span a face of the polytope $C(n, d)$. We assume that $t_1 < t_2 < .. < t_n$. Let $\{x(t_1), .., x(t_n)\}$ be the vertex set of $C(n, d)$.

Definition 1.4. Let $X \subset \{x(t_1), .., x(t_n)\}$. A component of X is a non-empty subset $Y = \{x(t_j), x(t_{j+1}), .., x(t_{k-1}), x(t_k)\}$ of X such that $x(t_{j-1}) \notin X$ for $j > 1$ and $x(t_{k+1}) \notin X$ for $k < n$.

A component Y is *proper* if $x(t_1)$ and $x(t_n)$ do not belong to the set $\{x(t_j), x(t_{j+1}), .., x(t_{k-1}), x(t_k)\}$. A component containing an odd number of vertices is called an *odd component.* The following theorem from [**B**] describes when a set of vertices span a face of a cyclic polytope.

THEOREM 1.5. *Let $C(n,d)$ be a cyclic polytope such that $t_1 < t_2 < .. < t_n$. Let X be a subset of $\{x(t_j), x(t_{j+1}), .., x(t_{k-1}), x(t_k)\}$ containing k points where $k \leq d$. Then X is the set of vertices of a $(k-1)$ face of $C(n,d)$ if and only if the number of odd components of X is at most $d-k$.*

The relation between cyclic polytopes and neighborliness is given by the following theorem that appears in [**B**].

THEOREM 1.6. *A cyclic polytope of type $C(n,d)$ is a simplicial k polytope for $k \leq [\frac{d}{2}]$.*

It is stated in [**B**] pg. 92 that the dual of a cyclic polytope is a dual neighborly polytope.

In what follows P is an n dimensional, *q-neighborly*, *simple convex polytope* unless otherwise stated.

DEFINITION 1.7. Let $F = \{F_1, .., F_m\}$ be the set of facets of P. For a fixed commutative ring R with unit we have

$$R(P) = R[v_1, .., v_m]/I = \langle v_{i_1} \cdots v_{i_k} \,|\, \bigcap_{j=1}^{k} F_{i_j} = \emptyset \rangle$$

where $|v_i| = 2$ are indexed by the facets and the ideal I is generated by square free monomials.

In the sequel we will be interested in $\mathbb{Z}(P)$. We denote by $|I|$ the cardinality of of the generating set of the ideal. There is a relation between $\mathbb{Z}(P)$ and $\mathbb{Z}(P^*)$. Given P and its polar P^*. Let K_P be the boundary of P^*, then $\mathbb{Z}(P) \cong \mathbb{Z}(K_P)$.

2. Toric Spaces and the Borel Space

Let P be as in the previous section; T^m is the m dimensional topological torus which is a product of circles indexed by the facets of P and BT^m the classifying space for T^m. For each co-dimension k face $F_k = \bigcap_{j=1}^{k} F_{\iota_j}$ there is a coordinate subgroup $T^{F_k} = S^1_{F_{\iota_1}} \times S^1_{F_{\iota_2}} \times \cdots \times S^1_{F_{\iota_k}}$.

DEFINITION 2.1. Let a space X be endowed with a torus action. Then the Borel space $B_T X$ is the identification space

$$ET^m \times X/\sim \; = ET^m \times_{T^m} X$$

where the equivalence relation is defined by: $(e,x) \sim (eg, g^{-1}x)$ for any $e \in ET^m$ and $x \in X$, $g \in T^m$

REMARK 2.2. $B_T P$ refers to $ET^m \times X/\sim$ which has the homotopy type as P when P is a simple convex polytope [**DJ**]. There is no ambiguity in writing $B_T P$ instead of $B_T X$-the Borel construction applied to a space. It is also shown in [**DJ**] that $H^*(B_T P) \cong \mathbb{Z}(P)$

Sometimes one refers to this as applying the Borel construction to X. One nice property of this definition is that it allows for one to move the torus action into the quotient. Another very important characteristic of the Borel space is the existence of the fibration $X \to ET^m \times_{T^m} X \to BT^m$.

We take as our definition of the *moment angle complex* [**BP**], [**DJ**]

Definition 2.3.

$$Z_P = T^m \times P/\sim$$

where $(g,p) \sim (h,q) \Leftrightarrow p = q \in F_k$ and $g^{-1}h \in T^{F_k}$

The reader should note that this definition carries over immediately when P is a simple polyhedral complex. However, it is not the case that Z_P is a manifold. T^m acts on Z_P [**BP**] from which the existence of the fibration $Z_P \to B_TP \to BT^m$ follows. In fact, Z_K exists for any simplicial complex K. The Borel construction applied to the corresponding moment angle complex gives the space $B_TZ_K = ET^m \times_{T^m} Z_K$ and the associated fibration $Z_K \to B_TZ_K \to BT^m$.

If K is a simpicial sphere then Z_K is a manifold. It should be noted that the moment angle complex behaves well with respect to products and other operations on the level of the polytope [**BP**]. $U(K)$ is the complex coordinate subspace arrangement complement associated to K. We now give its definition. First, we assume that K is an $n-1$ dimensional simplicial complex on a vertex set $\{v_1,..,v_m\}$ denoted by $[m]$. Let $\sigma = \{\iota_1,..,\iota_k\} \subset [m]$. A *coordinate subspace* of $\mathbb{C}^m$ is $L_\sigma = \{(z_1,..,z_m) \in \mathbb{C}^m | z_{\iota_1} = \cdots = z_{\iota_k} = 0\}$ [**BP**]. The dimension of $L_\sigma = m - |\sigma|$. An *arrangement* $A = \{L_1,..,L_j\}$ is *coordinate* if each L_i is coordinate. The *complex coordinate subspace arrangement* associated to K, $CA(K) = \{L_\sigma | \sigma \notin K\}$ and the *complex coordinate subspace arrangement complement* is $U(K) = \mathbb{C}^m \setminus \bigcup_{\sigma \notin K} L_\sigma$. In [**BP**] it is shown that there is a bijection between the set of simplicial complexes on $[m]$ and the set of coordinate subspace arrangement complements in $\mathbb{C}^m$. We list two examples taken from [**BP**].

Example 2.4. If $K = \partial\Delta^{m-1}$ then $U(K) = \mathbb{C}^m \setminus \{0\}$

and

Example 2.5. If K_P is dual to an m-gon then

$$U(K) = \mathbb{C}^m \setminus \bigcup_{i-j\neq 0,1\ mod\ (m)} \{z_i = z_j = 0\}$$

[**BP**] proved that $Z_K \subset U(K)$ and that there is an equivariant deformation retraction $U(K) \to Z_K$. In particular, any information concerning the homotopy of Z_K gives information about the homotopy of $U(K)$. It is clear from this theorem and the fibration: $Z_K \to B_TP \to BT^m$ that $\pi_*(U(K)) \cong \pi_*(B_TP)$ for $* > 2$. This will be essential in showing that the conjecture can not be true.

3. Homotopy Theory

In this section we recall the Hilton-Milnor theorem, one of the main theorems from [**A**] as well as some terminology that appears in the sequel.

Let A_k be the free nonassociative algebra on $\{x_1, x_2, \ldots, x_k\}$. If $w \in A$ is a monomial we define the weight of w, $W(w)$, to be the number of its factors and $a_i(w)$ be the number of times that the generator x_i appears in w. Following [**W**], we single out certain elements in A_k called basic products. We single out $\mathcal{A}_i(A_k)$ the basic products of weight i. We define $\mu : \mathbb{N} \to \mathbb{N}$ such that

$$\mu(n) = \begin{cases} 1 & n = 1 \\ (-1)^k & n = p_1 \cdots p_k,\ p_i \neq p_j, p_i \text{ prime},\ n > 1 \\ 0 & \text{otherwise} \end{cases}$$

THEOREM 3.1. *The number of elements in $\mathcal{A}_n(A_k)$ is given by the formula*

$$\frac{1}{n} \sum_{d|n} \mu(d) k^{\frac{n}{d}} \tag{3.1}$$

Let $X_1, X_2, \ldots, X_k$ be connected spaces with basepoint. For $w \in \mathcal{A}_*(A_k)$ a monomial we define

$$w(X_1, X_2, \ldots, X_k) = (X_1)^{a_1(w)} \wedge (X_2)^{a_2(w)} \wedge \ldots (X_k)^{a_k(w)}$$

Where $(X)^k$ is X smashed with itself k times.

THEOREM 3.2 (Hilton-Milnor). *There is a map*

$$h : J(X_1 \vee X_2 \vee \ldots \vee X_k) \to \prod_{w \in \mathcal{A}_*(A_k)} Jw(X_1, \ldots, X_k) \tag{3.2}$$

where J is the James reduced product. Moreover this map is a homotopy equivalence.

In the case $k = 16$ and $X_i = S^4$ for every $1 \leq i \leq 16$. The left side of (3.2) gives

$$J(\vee_{16} S^4) \simeq \Omega\Sigma(\vee_{16} S^4) \simeq \Omega(\vee_{16} S^5)$$

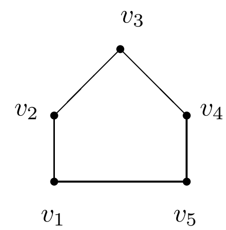
and the right side of (3.2) gives

$$\begin{aligned} \prod_{w \in \mathcal{A}_*(A_{16})} Jw(S^4, \ldots, S^4) &\simeq \prod_{w \in \mathcal{A}_*(A_{16})} J(S^{4a_1(w) + \cdots + 4a_k(w)}) \\ &\simeq \prod_{w \in \mathcal{A}_*(A_{16})} \Omega\Sigma S^{4(a_1(w) + \cdots + a_k(w))} \\ &\simeq \Omega \prod_{w \in \mathcal{A}_*(A_{16})} S^{(4W(w)+1)} \end{aligned}$$

The number of spheres of dimension $4W(w) + 1$ can be calculated using formula (3.1). All of this implies

$$\pi_*(\vee_{16} S^5) \cong \prod_{w \in \mathcal{A}_*(A_{16})} \pi_*(S^{(4W(w)+1)}) \tag{3.3}$$

We briefly summarize the notion of relations among relations that appears in [**A**]. Consider the following 2 dimensional simple convex polytope.

$P \quad =$ 

It is clear that the face ring $\mathbb{Z}(P) \cong \mathbb{Z}[v_1, .., v_5]/\langle v_1v_3, v_2v_4, v_3v_5, v_4v_1, v_5v_2\rangle$. Resolving the ideal in the category of free $\mathbb{Z}$ algebras produces a resolution:

$$R^* \xrightarrow{\iota^*} C^* \xrightarrow{p^*} \mathbb{Z}(P)$$

where p^* is the induced map in cohomology coming from $p : B_TP \to BT^m$, R^* is the free algebra generated by $ker\ p^*$ and C^* is the free algebra. More explicitly, for this example we have

$$\mathbb{Z}[x_1, .., x_5] \xrightarrow{\iota^*} \mathbb{Z}[v_1, .., v_5] \longrightarrow \mathbb{Z}(P)$$

where ι^* sends the x_i to the monomials that generate the ideal I. For example, $\iota^*(x_1) = v_1v_3$ and $\iota^*(x_2) = v_2v_4$.

Definition 3.3. Let $\overline{I}$ and I' be two multi-indexes such that $\overline{I} \neq I'$ and $\overline{I}, I' \subset S = \{\iota_1, .., \iota_m\}$. Suppose $i \neq j$. We call a relation of the form:

$$\iota^*(x_i) \prod_{k \in \overline{I}} v_k - \iota^*(x_j) \prod_{k' \in I'} v_{k'} = 0$$

in C^* a relation among relations. We denote such a relation by $\overline{R}$.

In the example above, a relation among relations is $\overline{R} = \iota^*(x_1)v_2v_4 - \iota^*(x_2)v_1v_3$. The degree of this relation among relations denoted by $|\overline{R}|$ is 8. $\Re_{min}$ is a relation among relations of the smallest degree. In general, $\Re_{min}$ is not unique as one can readily verify from the example above. In [**A**] the BP analogue of this procedure was carried out with the idea of setting up the Unstable Adams Novikov Spectral Sequence through a range. Let $r_j \in I$.

Theorem 3.4. [**A**] *Suppose P is a q neighborly, n dimensional simple polyhedral complex and I is the ideal in the face ring $\mathbb{Z}(P)$. Let p be a prime number, m be the number of facets of P.*

$\pi_{s-1}(B_TP)_{(p)} = \pi_{s-1}((\bigvee_{r_j \in I} S^{|r_j|-1}))_{(p)}$ *for* $s \leq |\Re_{min}| - 1$

and $\pi_2(B_TP) = \mathbb{Z}^{\oplus m}$

4. Main Result

In [**BM**], it is shown that when P is the cyclic ploytope $C(8,4)$ the cohomology ring of the moment angle complex is

$$H^*(Z_P) \cong H^*((\sharp_{16}S^5 \times S^7)\sharp(\sharp_{15}S^6 \times S^6))$$

It was stated in [**BM**] that it was unknown whether or not Z_P is diffeomorphic to the connected sum $(\sharp_{16}S^5 \times S^7)\sharp(\sharp_{15}S^6 \times S^6)$. We compute one of the higher homotopy groups of this connected sum and apply Theorem 3.4 to compare it to the same higher homotopy group of B_TP. We need a few preliminary results.

LEMMA 4.1. For $P = C(8,4)$, the face ring $\mathbb{Z}(P) = \mathbb{Z}[v_1, .., v_8]/I$ where I is generated by the square free monomials:

- $v_1v_3v_5$, $v_1v_3v_6$, $v_1v_3v_7$, $v_1v_4v_6$
- $v_1v_4v_7$, $v_1v_5v_7$, $v_2v_4v_6$, $v_2v_4v_7$
- $v_2v_4v_8$, $v_2v_5v_7$, $v_2v_5v_8$, $v_2v_6v_8$
- $v_3v_5v_7$, $v_3v_5v_8$, $v_3v_6v_8$, $v_4v_6v_8$

PROOF. This follows from an application of Theorem 1.5 applied to the cyclic polytope $P = C(4,8)$.

□

REMARK 4.2. From this explicit description of the face ring one readily sees that $|\Re_{min}| = 8$. It is necessary to know what the range is when applying 3.4. The reader should note that the input for 3.4 consists of $|\Re_{min}|$, which in many cases depends on computing the face ring $\mathbb{Z}(P)$ explicitly.

We compute the integral homology of $(\sharp_{16}S^5 \times S^7)\sharp(\sharp_{15}S^6 \times S^6)$. Let $T_{m,n} = S^m \times S^n$

LEMMA 4.3. Let $M = (\sharp_{16}T_{5,7})\sharp(\sharp_{15}T_{6,6})$. Then

$$\widetilde{H}_k(M) = \begin{cases} \mathbb{Z}^{30} & k = 6 \\ \mathbb{Z}^{16} & n = 5,7 \\ \mathbb{Z} & n = 12 \\ 0 & \text{otherwise} \end{cases}$$

PROOF. Let $U \subseteq T_{m,n}$ be a contractible open set. Then by the Künneth theorem and excision we have

$$\widetilde{H}_k(T_{m,n} - U) = \begin{cases} \mathbb{Z} & k = m,n,\ m \neq n \\ \mathbb{Z}^2 & k = m = n \\ 0 & \text{otherwise} \end{cases}$$

It is clear that there exists a cofibration

$$(T_{m,n} - U) \to \sharp_i T_{m,n} \to \sharp_i T_{m,n}/(T_{m,n} - U) \backsimeq \sharp_{(i-1)}T_{m,n} \tag{4.1}$$

Successive applications of the long exact sequence in homology induced from (4.1) yields

$$\widetilde{H}_k(\sharp_{16}T_{5,7}) = \begin{cases} \mathbb{Z}^{16} & k = 5,7 \\ \mathbb{Z} & k = 12 \\ 0 & \text{otherwise} \end{cases}$$

and

$$\widetilde{H}_k(\sharp_{15}T_{6,6}) = \begin{cases} \mathbb{Z}^{30} & k = 6 \\ \mathbb{Z} & k = 12 \\ 0 & \text{otherwise} \end{cases}$$

Now we use the cofibration

$$((\sharp_{15}T_{6,6}) - U) \to M \to M/(\sharp_{15}T_{6,6} - U) \backsimeq \sharp_{16}T_{5,7}$$

to obtain a long exact sequence

$$\cdots \to \widetilde{H}_k((\sharp_{15}T_{6,6}) - U) \to \widetilde{H}_k(M) \to \widetilde{H}_k(\sharp_{16}T_{5,7}) \to \cdots$$

This long exact sequence in homology immediately gives $\widetilde{H}_5(M) \cong \mathbb{Z}^{16}$ and

$$0 \to \widetilde{H}_7(M) \to \mathbb{Z}^{16} \underset{j}{\to} \mathbb{Z}^{30} \to \widetilde{H}_6(M) \to 0 \tag{4.2}$$

Since M is a compact, closed, orientable, manifold of dimension 12, then $H_{12}(M) \cong \mathbb{Z}$ and by Poincaré duality there is an isomorphism $H^7(M) \cong H_5(M) \cong \mathbb{Z}^{16}$. Since there is no torsion we obtain the isomorphism $H_7(M) \cong \mathbb{Z}^{16}$ and hence the map $j : \mathbb{Z}^{16} \to \mathbb{Z}^{30}$ must factor through zero, giving us $H_6(M) \cong \mathbb{Z}^{30}$.

□

Now the main result:

THEOREM 4.4. *For $P = C(4,8)$, the moment angle complex Z_P is not homotopy equivalent to $(\sharp_{16}S^5 \times S^7)\sharp(\sharp_{15}S^6 \times S^6)$.*

PROOF. If Z_P were homotopy equivalent to $\sharp_{16}(S^5 \times S^7)\sharp(\sharp_{15}S^6 \times S^6)$, then by Lemma 4.1 and Theorem 3.4 we have

$$\pi_6(\bigvee_{16} S^5)_{(2)} \cong \pi_6((\sharp_{16}S^5 \times S^6)\sharp(\sharp_{15}S^6 \times S^6))_{(2)}$$

We use Serre's method to calculate $\pi_6(M)$. Let F be the homotopy fiber of the map $M \to \prod_{16} K(\mathbb{Z},5)$. There exists a fibration $\prod_{16} K(\mathbb{Z},4) \to F \to M$. It is known that $H^*(K(\mathbb{Z},4);\mathbb{Q}) \cong \mathbb{Q}[x]$ where $|x| = 4$ [**H**]. It follows from a Serre Spectral Sequence calculation and the universal coefficients theorem that $\widetilde{H}_6(F;\mathbb{Q}) \cong \mathbb{Q}^{30}$. Since F is 5-connected, application of the Hurewicz theorem yields the isomorphisms

$$\pi_6(M) \otimes \mathbb{Q} \cong \pi_6(F) \otimes \mathbb{Q} \cong H_6(F) \otimes \mathbb{Q} \cong H_6(F;\mathbb{Q}) \cong \mathbb{Q}^{30}$$

We now can compare this result with $\pi_6(\vee_{16}S^5)_{(2)}$. Using (3.1) and (3.3) we know that $\pi_*(\vee_{16}S^5) \cong \pi_*(\prod_{16} S^5) \times \pi_*(\prod_k S^k)$ where $k > 8$. It suffices to check $\pi_6(\prod_{16} S^5)$. But this group is all torsion. This implies that the group $\pi_6(\vee_{16}S^5)\otimes\mathbb{Q}$ is the trivial group.

□

Acknowledgements

The first author would like to thank Taras Panov who pointed out during the International Conference on Toric Topology in Osaka Japan, 2006 that the results in [**A**] might be applicable with respect to this conjecture of Bosio and Meersseman. Both authors thank the referee for many helpful suggestions that improved the exposition of the paper.

References

[A] David Allen, *On the Homotopy Groups of Toric Spaces,* submitted.

[BM] Frederic Bosio and Laurent Meersseman, *Real quadrics in $\mathbb{C}^n$, complex manifolds and convex polytopes.* Acta Math. 197 (2006), no. 1, 53–127.

[B] A. Bronsted, *An Introduction to Convex Polytopes ,* Springer-Verlag, New York, 1983.

[BP] V.M. Buchstaber and T.E. Panov, *Torus actions and their applications in Topology and Combinatorics,* University Lecture Seriers, AMS (2002).

[H] Allen Hatcher, *Algebraic Topology*, Cambridge University Press.

[DJ] M. Davis and T. Januszkiewicz, *Convex polytopes, Coxter orbifolds and torus actions,* Duke Math. Journal 62 no. 2, 417-451.

[W] George W. Whitehead, *Elements of Homotopy Theory*, Graduate Texts in Mathematics, Springer.

Department of Mathematics Iona College, New Rochelle, 10801
E-mail address: dallen@iona.edu

Department of Mathematics, University of Puerto Rico, PO BOX 23355 San Juan, Puerto Rico 00931-3355
E-mail address: jluz@cnnet.upr.edu